A Pair of Three-Step Hybrid Block Methods for the Solutions of Linear and Nonlinear First-Order Systems

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Abstract
For first-order systems that are either linear or nonlinear, this study derives two three-step hybrid block approaches. Applying the collocation and interpolation approach, with power series as the basis function, allows for the derivation to be carried out. By include a single off-grid point and two off-grid points inside the three-step integration interval, respectively, the first and second three-step hybrid block approaches are generated. To evaluate their accuracy and efficiency, the obtained techniques were used to certain linear and nonlinear first-order systems.Based on the outcomes, it is clear that the three-step hybrid block technique including two off-grid spots outperformed its one-off-grid counterpart. We also found that the two derived approaches outperformed the current methods we compared them against. This was evident from the findings. We dug further into the fundamental features of the obtained approaches. The zero-stability, consistence, convergence, and absolute stability regions are all characteristics that fall under this category.

Keywords: This is a first-order hybrid linear nonlinear off-grid three-step system.

I. Introduction

In this paper, we shall derive a pair of three-step methods for the solution of linear and nonlinear first order systems of the form,

$$y'(x) = f(x, y),\ y(x_0) = y_0 \qquad (1)$$

where $f : \Re \times \Re^{2q} \rightarrow \Re^{q}$; $y, y \in \Re^{q}$, and q is the dimension of the system. The function f is assumed to satisfy the Lipschitz condition stated in the Theorem below.

f (x,y) bea function,defined andcontinuousfor allpoints(x, y)intheregionDdefinedby$a \leq x \leq b, -\infty < y < \infty$),aandbfinite,andletthereexis taconstantLsuchthat,foreveryx,y,y* such that (x, y) and f (x, y*) are both in D,

$$\left|f(x, y) - f(x, y^*)\right| \leq L\left|y - y^*\right| \qquad (2)$$

Then, if η is any given number, there exists a unique solution y(x) of the initial value problem (1), where y(x) is continuous and differentiable for all (x, y) in D . The requirement (2) is known as a Lipschitz condition and the constant L as a Lipschitz constant.

A lot of algorithms and methods have been proposed by scholars for the solution of first-order stiff systems of the form (1), these authors among others include [2]-[9]. However, it is important to state that these methods have some setbacks ranging from small convergence/implementation region to inefficiency in terms of accuracy. In view of these setbacks, we are motivated to formulate a pair of three-step hybrid block methods that will address some of these setbacks. The proposed methods have the advantage of generating simultaneous numerical approximations at different grid points within the interval of integration. Another advantage of the methods is that they are less expensive in terms of the number of function evaluations compared to the conventional linear multistep and the Runge-Kuttamethods. They also preserve the traditional advantage of one-step methods of being self-starting and permitting easy change of step-size during integration, [10].

II. DERIVATIONOFTHE THREE-STEPHYBRIDBLOCK METHODS

Inthis section,apairofthree-step hybridblock methodsof the form,

$$A^{(0)}Y_m = Ey_n + hdf(y_n) + hbf(Y_m) \quad (3)$$

Where A(0) , E, d and b are (r -1) x (r -1)matrices shall be derived for the solution of linear and nonlinear first order systems of the form (1). To achieve this, we approximate the exact solution y(x) to (1) by assuming an approximate solution Y (x) in the form,

$$Y(x) = \sum_{i=0}^{r+s-1} p_i \omega_i(x) \quad (4)$$

whererandsarethenumbersofcollocationandinterpolationpointsrespectively, $x \in [x_0, x_n]$, p_i, are undetermined coefficients that must be obtained and $\omega_i(x)$ are basis polynomial function of degree r + s -1 .

A. Three-StepHybridBlock Methodwith OneOff-Grid Point

Toderivethethree-stephybridblockmethodwithoneoff-gridpoint,wecarryoutinterpolationat x_{n+s}, s= ½ and collocation at x_{n+r}, r= 0,1/2,1,2,3 as follows,

$$\sum_{j=0}^{5} p_j x_{n+s}^{j} = y_{n+s},\ s = \frac{1}{2} \quad (5)$$

$$\sum_{j=0}^{5} j p_j x_{n+r}^{j-1} = f_{n+r},\ r = 0, \frac{1}{2}, 1, 2, 3 \quad (6)$$

Equations(5)and(6) givesasystemof nonlinear equationofthe form

$$XA=U \qquad (7)$$

Where,

$$A = \left[a_0\ a_1\ a_2\ a_3\ a_4\ a_5\right]^T,\quad U = \left[y_n\ f_n\ f_{n+\frac{1}{2}}\ f_{n+1}\ f_{n+2}\ f_{n+3}\right]^T$$

$$X = \begin{bmatrix} 1 & x_{n+\frac{1}{2}} & x_{n+\frac{1}{2}}^2 & x_{n+\frac{1}{2}}^3 & x_{n+\frac{1}{2}}^4 & x_{n+\frac{1}{2}}^5 \\ 0 & 1 & 2x_n & 3x_n^2 & 4x_n^3 & 5x_n^4 \\ 0 & 1 & 2x_{n+\frac{1}{2}} & 3x_{n+\frac{1}{2}}^2 & 4x_{n+\frac{1}{2}}^3 & 5x_{n+\frac{1}{2}}^4 \\ 0 & 1 & 2x_{n+1} & 3x_{n+1}^2 & 4x_{n+1}^3 & 5x_{n+1}^4 \\ 0 & 1 & 2x_{n+2} & 3x_{n+2}^2 & 4x_{n+2}^3 & 5x_{n+2}^4 \\ 0 & 1 & 2x_{n+3} & 3x_{n+3}^2 & 4x_{n+3}^3 & 5x_{n+3}^4 \end{bmatrix}$$

Solving thesystem ofnonlinearequation by Gauss elimination method forthe p_j's, j =0(1)5and substituting back into (4) gives a continuous three-step hybrid block method with one off-grid point of the form,

$$Y(x) = \alpha_{\frac{1}{2}}(x) y_{n+\frac{1}{2}} + h\left(\sum_{j=0}^{3} \beta_k(x) f_{n+k} + \beta_{\frac{1}{2}}(x) f_{n+\frac{1}{2}}\right) \quad (8)$$

where

$$\left.\begin{aligned}
&\alpha_{\frac{1}{2}}(x)=1\\
&\beta_0(x)=\frac{1}{5760}\left(384t^5-3120t^4+8960t^3-11040t^2+5760t-1057\right)\\
&\beta_{\frac{1}{2}}(x)=-\frac{1}{225}\left(48t^5-360t^4+880t^3-720t^2+91\right)\\
&\beta_1(x)=-\frac{1}{1920}\left(384t^5-2640t^4+5440t^3-2880t^2+193\right)\\
&\beta_2(x)=-\frac{1}{5760}\left(384t^5-2160t^4+3200t^3-1440t^2+83\right)\\
&\beta_3(x)=\frac{1}{28800}\left(384t^5-1680t^4+2240t^3-960t^2+53\right)
\end{aligned}\right\} \quad (9)$$

andt isgiven by

$$t=\frac{x-x_n}{h} \quad (10)$$

Evaluating (8)att=½, 1,2,3 givesthenewdiscretethree-stephybridblock method with oneoff- grid point of the form (3) as

$$\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}y_{n+\frac{1}{2}}\\y_{n+1}\\y_{n+2}\\y_{n+3}\end{bmatrix}=\begin{bmatrix}0&0&0&1\\0&0&0&1\\0&0&0&1\\0&0&0&1\end{bmatrix}\begin{bmatrix}y_{n-\frac{1}{2}}\\y_{n-1}\\y_{n-2}\\y_n\end{bmatrix}+h\begin{bmatrix}0&0&0&\frac{1057}{5760}\\0&0&0&\frac{59}{360}\\0&0&0&\frac{11}{45}\\0&0&0&\frac{3}{40}\end{bmatrix}\begin{bmatrix}f_{n-\frac{1}{2}}\\f_{n-1}\\f_{n-2}\\f_n\end{bmatrix}$$
$$+h\begin{bmatrix}\frac{91}{225}&-\frac{193}{1920}&\frac{83}{5760}&-\frac{53}{28800}\\\frac{152}{225}&\frac{19}{120}&\frac{1}{360}&-\frac{1}{1800}\\\frac{64}{225}&\frac{16}{15}&\frac{19}{45}&-\frac{4}{225}\\\frac{24}{25}&\frac{9}{40}&\frac{57}{40}&\frac{63}{200}\end{bmatrix}\begin{bmatrix}f_{n+\frac{1}{2}}\\f_{n+1}\\f_{n+2}\\f_{n+3}\end{bmatrix} \quad (11)$$

B. Three-StepHybridBlockMethodwithTwoOff-GridPoints

For three-step hybrid block method with two off-grid points, we interpolate at x_{n+s}, s=3/2 and collocate at xn+r, r = 0, ½, 1, 3/2, 2, 3. This gives,

$$\sum_{j=0}^{6} p_j x_{n+s}^{j} = y_{n+s},\ s = \frac{3}{2} \tag{12}$$

$$\sum_{j=0}^{6} j p_j x_{n+r}^{j-1} = f_{n+r},\ r = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, 3 \tag{13}$$

Equations(12)and (13) will togethergiveasystem nonlinearequation oftheform(7),where

$$A = \begin{bmatrix} a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \end{bmatrix}^T,\quad U = \begin{bmatrix} y_n & f_n & f_{n+\frac{1}{2}} & f_{n+1} & f_{n+\frac{3}{2}} & f_{n+2} & f_{n+3} \end{bmatrix}^T$$

$$X = \begin{bmatrix} 1 & x_{n+\frac{3}{2}} & x^2_{n+\frac{3}{2}} & x^3_{n+\frac{3}{2}} & x^4_{n+\frac{3}{2}} & x^5_{n+\frac{3}{2}} & x^6_{n+\frac{3}{2}} \\ 0 & 1 & 2x_n & 3x_n^2 & 4x_n^3 & 5x_n^4 & 6x_n^5 \\ 0 & 1 & 2x_{n+\frac{1}{2}} & 3x^2_{n+\frac{1}{2}} & 4x^3_{n+\frac{1}{2}} & 5x^4_{n+\frac{1}{2}} & 6x^5_{n+\frac{1}{2}} \\ 0 & 1 & 2x_{n+1} & 3x^2_{n+1} & 4x^3_{n+1} & 5x^4_{n+1} & 6x^5_{n+1} \\ 0 & 1 & 2x_{n+\frac{3}{2}} & 3x^2_{n+\frac{3}{2}} & 4x^3_{n+\frac{3}{2}} & 5x^4_{n+\frac{3}{2}} & 6x^5_{n+\frac{3}{2}} \\ 0 & 1 & 2x_{n+2} & 3x^2_{n+2} & 4x^3_{n+2} & 5x^4_{n+2} & 6x^5_{n+2} \\ 0 & 1 & 2x_{n+3} & 3x^2_{n+3} & 4x^3_{n+3} & 5x^4_{n+3} & 6x^5_{n+3} \end{bmatrix}$$

Similarly, solvingthesystemofnonlinearequationbyGausseliminationmethodfor thep_j's,j =0(1)6 andsubstitutingbackinto(4)givesacontinuousthree-stephybridmethodwith two off-grid points of the form,

$$Y(x) = \alpha_{\frac{3}{2}}(x) y_{n+\frac{3}{2}} + h\left(\sum_{j=0}^{3} \beta_k(x) f_{n+k} + \beta_{\frac{1}{2}}(x) f_{n+\frac{1}{2}} + \beta_{\frac{3}{2}}(x) f_{n+\frac{3}{2}} \right) \tag{14}$$

where

$$\left.\begin{aligned}
&\alpha_{\frac{3}{2}}(x)=1\\
&\beta_0(x)=-\frac{1}{17280}\left(640t^6-6144t^5+22800t^4-41600t^3+38880t^2-17280t+2781\right)\\
&\beta_{\frac{1}{2}}(x)=\frac{1}{360}\left(64t^6-576t^5+1920t^4-2880t^3+1728t^2-243\right)\\
&\beta_1(x)=-\frac{1}{1920}\left(640t^6-5376t^5+16080t^4-20160t^3+8640t^2+729\right)\\
&\beta_{\frac{3}{2}}(x)=\frac{1}{1080}\left(320t^6-2496t^5+6720t^4-7360t^3+2880t^2-351\right)\\
&\beta_2(x)=-\frac{1}{5760}\left(640t^6-4608t^5+11280t^4-11520t^3+4320t^2-243\right)\\
&\beta_3(x)=\frac{1}{17280}\left(128t^6-768t^5+1680t^4-1600t^3+576t^2-27\right)
\end{aligned}\right\} \quad (15)$$

andtasdefinedin(10).Evaluating(14)att=½,1,3/2,2,3givesthenewdiscretethree-step hybrid block method with two off-grid points of the form (3) as,

$$\begin{bmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{bmatrix}\begin{bmatrix}y_{n+\frac{1}{2}}\\y_{n+1}\\y_{n+\frac{3}{2}}\\y_{n+2}\\y_{n+3}\end{bmatrix}=\begin{bmatrix}0&0&0&0&1\\0&0&0&0&1\\0&0&0&0&1\\0&0&0&0&1\\0&0&0&0&1\end{bmatrix}\begin{bmatrix}y_{n-\frac{1}{2}}\\y_{n-1}\\y_{n-\frac{3}{2}}\\y_{n-2}\\y_n\end{bmatrix}+h\begin{bmatrix}0&0&0&0&\frac{959}{5760}\\0&0&0&0&\frac{169}{1080}\\0&0&0&0&\frac{103}{640}\\0&0&0&0&\frac{7}{45}\\0&0&0&0&\frac{11}{40}\end{bmatrix}\begin{bmatrix}f_{n-\frac{1}{2}}\\f_{n-1}\\f_{n-\frac{3}{2}}\\f_{n-2}\\f_n\end{bmatrix}$$

$$+h\begin{bmatrix}\frac{35}{72}&-\frac{487}{1920}&\frac{49}{360}&-\frac{211}{5760}&\frac{1}{640}\\\frac{32}{45}&\frac{11}{120}&\frac{8}{135}&-\frac{7}{360}&\frac{1}{1080}\\\frac{27}{40}&\frac{243}{640}&\frac{13}{40}&-\frac{27}{640}&\frac{1}{640}\\\frac{32}{45}&\frac{4}{15}&\frac{32}{45}&\frac{7}{45}&0\\0&\frac{81}{40}&-\frac{8}{5}&\frac{81}{40}&\frac{11}{40}\end{bmatrix}\begin{bmatrix}f_{n+\frac{1}{2}}\\f_{n+1}\\f_{n+\frac{3}{2}}\\f_{n+2}\\f_{n+3}\end{bmatrix} \quad (16)$$

III. ANALYSISOF THE THREE-STEPHYBRIDBLOCK METHODS

In this section, some basic properties of the pair of three-step hybrid block methods shall be analyzed. These properties among others include the order, consistency, zero-stability, convergence and stability region.

A. Order andErrorConstantsof theThree-StepHybridBlock Methods

ThelinearoperatorL$\{y(x);h\}$ofthethree-stepmethodoftheform(4)isdefinedas,

$$L\{y(x);h\} = A^{(0)}Y_m - Ey_n - hdf(y_n) - hbf(Y_m) \tag{17}$$

TakingtheTaylorseriesexpansionof(17) andcomparingthecoefficientsofhgives,

$$\begin{aligned} L\{y(x);h\} &= c_0 y(x) + c_1 hy'(x) + c_2 h^2 y''(x) \\ &+ \ldots + c_p h^p y^p(x) + c_{p+1} h^{p+1} y^{p+1}(x) + \ldots \end{aligned} \tag{18}$$

Definition1[11]

Amethodissaidtobeoforderpifpisthelargestpositiveintegerforwhich $\bar{c}_0 = \bar{c}_1 = \bar{c}_2 = \ldots = \bar{c}_p = 0, \bar{c}_{P+1} \neq 0.$
Theterm c p1 is called the □ error constant of the method. Sufficeto say thattheorderofamethodquantifiestherateofconvergenceofanumericalapproximationofa differentialequationtothatoftheexactsolution.Ontheotherhand,theerrorconstantisthe accumulatederrorwhentheorderofamethodhasbeencomputed.

Therefore, the Taylor series expansion of the three-step hybrid block method with one off-grid point is,

$$\begin{bmatrix} \sum_{j=0}^{\infty} \frac{\left(\frac{1}{2}\right)^j}{j!} y_n^j - y_n - \frac{1057}{5760} hy_n' - \sum_{j=0}^{\infty} \frac{h^{j+1}}{j!} y_n^{j+1} \left\{ \frac{91}{225}\left(\frac{1}{2}\right)^j - \frac{193}{1920}(1)^j + \frac{83}{5760}(2)^j - \frac{53}{28800}(3)^j \right\} \\ \sum_{j=0}^{\infty} \frac{(1)^j}{j!} y_n^j - y_n - \frac{59}{360} hy_n' - \sum_{j=0}^{\infty} \frac{h^{j+1}}{j!} y_n^{j+1} \left\{ \frac{152}{225}\left(\frac{1}{2}\right)^j + \frac{19}{120}(1)^j + \frac{1}{360}(2)^j - \frac{1}{1800}(3)^j \right\} \\ \sum_{j=0}^{\infty} \frac{(2)^j}{j!} y_n^j - y_n - \frac{11}{45} hy_n' - \sum_{j=0}^{\infty} \frac{h^{j+1}}{j!} y_n^{j+1} \left\{ \frac{64}{225}\left(\frac{1}{2}\right)^j + \frac{16}{15}(1)^j + \frac{19}{45}(2)^j - \frac{4}{225}(3)^j \right\} \\ \sum_{j=0}^{\infty} \frac{(3)^j}{j!} y_n^j - y_n - \frac{3}{40} hy_n' - \sum_{j=0}^{\infty} \frac{h^{j+1}}{j!} y_n^{j+1} \left\{ \frac{24}{25}\left(\frac{1}{2}\right)^j + \frac{9}{40}(1)^j + \frac{57}{40}(2)^j + \frac{63}{200}(3)^j \right\} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{19}$$

Thus, $\bar{c}_0 = \bar{c}_1 = \bar{c}_2 = \bar{c}_3 = \bar{c}_4 = \bar{c}_5 = 0;$ implyingthattheorderofthethree-stepmethodwithoneoff-grid point is*p* $=[5555]^T$. That is, the method is of uniform order 5.

The error constant is given by $\left[6.3802\times10^{-4} \quad 2.7778\times10^{-4} \quad 3.3333\times10^{-3} \quad -7.5000\times10^{-3}\right]^T$

Similarly, the Taylor series expansion of the three-step hybrid block method with two off-grid points is given by,

$$\begin{bmatrix} \sum_{j=0}^{\infty}\frac{\left(\frac{1}{2}\right)^j}{j!}y_n^j - y_n - \frac{959}{5760}hy_n' - \sum_{j=0}^{\infty}\frac{h^{j+1}}{j!}y_n^{j+1}\left\{\frac{35}{72}\left(\frac{1}{2}\right)^j - \frac{487}{1920}(1)^j + \frac{49}{360}\left(\frac{3}{2}\right)^j - \frac{211}{5760}(2)^j + \frac{1}{640}(3)^j\right\} \\ \sum_{j=0}^{\infty}\frac{(1)^j}{j!}y_n^j - y_n - \frac{169}{1080}hy_n' - \sum_{j=0}^{\infty}\frac{h^{j+1}}{j!}y_n^{j+1}\left\{\frac{32}{45}\left(\frac{1}{2}\right)^j + \frac{11}{120}(1)^j + \frac{8}{135}\left(\frac{3}{2}\right)^j - \frac{7}{360}(2)^j + \frac{1}{1080}(3)^j\right\} \\ \sum_{j=0}^{\infty}\frac{\left(\frac{3}{2}\right)^j}{j!}y_n^j - y_n - \frac{103}{640}hy_n' - \sum_{j=0}^{\infty}\frac{h^{j+1}}{j!}y_n^{j+1}\left\{\frac{27}{40}\left(\frac{1}{2}\right)^j + \frac{243}{640}(1)^j + \frac{13}{40}\left(\frac{3}{2}\right)^j - \frac{27}{640}(2)^j + \frac{1}{640}(3)^j\right\} \\ \sum_{j=0}^{\infty}\frac{(2)^j}{j!}y_n^j - y_n - \frac{7}{45}hy_n' - \sum_{j=0}^{\infty}\frac{h^{j+1}}{j!}y_n^{j+1}\left\{\frac{32}{45}\left(\frac{1}{2}\right)^j + \frac{4}{15}(1)^j + \frac{32}{45}\left(\frac{3}{2}\right)^j + \frac{7}{45}(2)^j + 0(3)^j\right\} \\ \sum_{j=0}^{\infty}\frac{(3)^j}{j!}y_n^j - y_n - \frac{11}{40}hy_n' - \sum_{j=0}^{\infty}\frac{h^{j+1}}{j!}y_n^{j+1}\left\{0\left(\frac{1}{2}\right)^j + \frac{81}{40}(1)^j - \frac{8}{5}\left(\frac{3}{2}\right)^j + \frac{81}{40}(2)^j + \frac{11}{40}(3)^j\right\} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (20)$$

Thus, $\bar{c}_0 = \bar{c}_1 = \bar{c}_2 = \bar{c}_3 = \bar{c}_4 = \bar{c}_5 = \bar{c}_6 = \bar{0}$ implyingthattheorderofthethree-stepmethodwithtwooff-grid points is $p = [6 \quad 6 \quad 6 \quad 6 \quad 6]^T$. That is, the method is of uniform order 6. It's error constant is given by

$$\left[-1.3589\times10^{-4} \quad -9.0939\times10^{-5} \quad -1.2556\times10^{-4} \quad -6.6138\times10^{-5} \quad -2.0089\times10^{-3}\right]^T$$

B. Consistencyof theThree-StepHybrid Block Methods

Recall that a method is consistent if it has order p $\geq$ 1, [11]. Thus, the three-step hybrid block method with one off-grid point (11) is consistent since it is of uniform order p = 5 $\geq$ 1. Also, the three-step hybrid block method with two off-grid points (16) is consistent since it is of uniform order p = 6 $\geq$ 1. Consistency controls the magnitude of the local truncation error committed at each stage of the computation.

C. Zero-StabilityoftheThree-StepHybridBlockMethods

Definition 2 [10]

Ablockmethodissaidtobezero-stable,iftherootszs,s □ 1,2,...,kofthefirstcharacteristic polynomial *p* (z) defined by *p*(z) = det(z$A^{(0)}$ - E) satisfies $|z_s| \leq 1$ and every root satisfying $|z_s| = 1$ have multiplicity not exceeding the order of the differential equation.

Forthethree-stepmethod with oneoff-gridpoint (11), thefirst characteristicpolynomial is given by,

$$\rho(z)=\left|z\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}-\begin{bmatrix}0&0&0&1\\0&0&0&1\\0&0&0&1\\0&0&0&1\end{bmatrix}\right|=\begin{vmatrix}z&0&0&-1\\0&z&0&-1\\0&0&z&-1\\0&0&0&z-1\end{vmatrix}=z^3(z-1)$$

Thus, solving for z in

$$z^3(z-1)=0 \tag{21}$$

Gives z1 = z2 = z3 = 0 and z4 = 1. Hence, the three-step hybrid block method with one off-grid point is zero-stable. Applying the same procedure for the three-step hybrid block method with two off-grid points (16), we obtain z1 = z2 = z3 = z4 = 0 and z5 = 1, implying that it is also is zero-stable.

D. ConvergenceoftheThree-StepHybridBlockMethods

Theorem 2 [10]

The necessary and sufficient conditions for the linear multistep method to be convergent are that it be consistent and zero-stable.

Therefore,thetwothree-stepmethodsderivedin(11)and(16)arebothconvergentsincetheyare consistent and zero-stable, [10].

E. RegionsofAbsoluteStabilityoftheThree-Step HybridBlock Methods

Applyingtheboundarylocusmethod,thestabilitypolynomialofthethree-stephybridblock method with one off-grid point (11) is given by,

$$\bar{h}(w)=-h^4\left(\frac{1}{8}w^3-\frac{1}{40}w^4\right)-h^3\left(\frac{23}{120}w^4+\frac{67}{120}w^3\right)+h^2\left(\frac{7}{10}w^4-\frac{13}{10}w^3\right) \tag{22}$$
$$-h\left(\frac{13}{10}w^4+\frac{17}{10}w^3\right)+w^4-w^3$$

Therefore,theregionofabsolutestabilityofthethree-stephybridblockmethodwithoneoff-grid point is shown in Fig. 1.

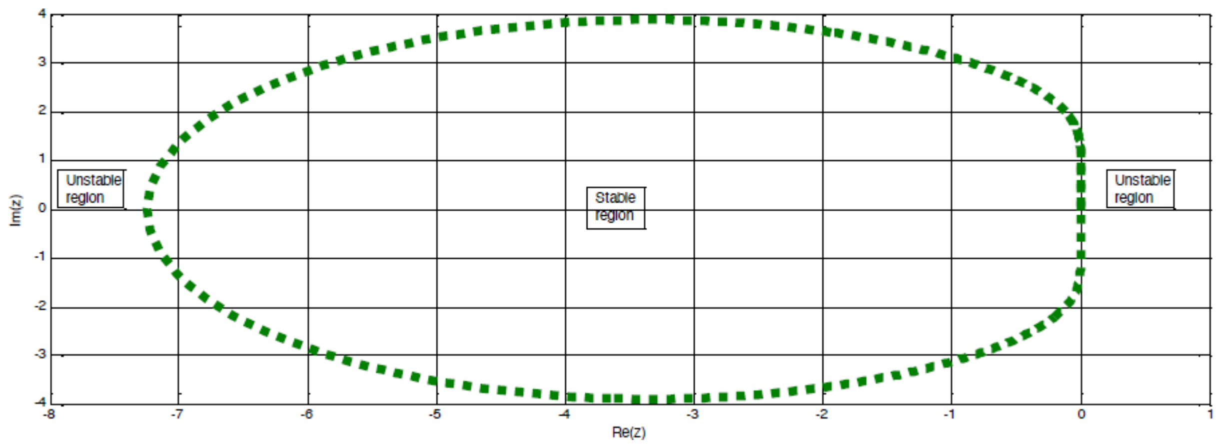

Fig.1. Stabilityregion ofthethree-stephybrid blockmethod withoneoff-grid point.

The stability region obtained in Fig. 1 is A-stable. Note that the stability region is the interior of the curve.

On the other hand, the stability polynomial of the three-step hybrid block method with two off-grid points (16) is given by,

$$\bar{h}(w) = -\left(\frac{1}{160}w^5 + \frac{1}{32}w^4\right) + h^4\left(\frac{9}{160}w^5 - \frac{29}{160}w^4\right) - h^3\left(\frac{13}{48}w^5 + \frac{29}{48}w^4\right) + h^2\left(\frac{19}{24}w^5 - \frac{31}{24}w^4\right) - h\left(\frac{4}{3}w^5 + \frac{5}{3}w^4\right) + w^5 - w^4 \quad (23)$$

Thus,theregionofabsolutestabilityofthethree-stephybridblockmethodwithtwooff-grid points is shown in Fig. 2.

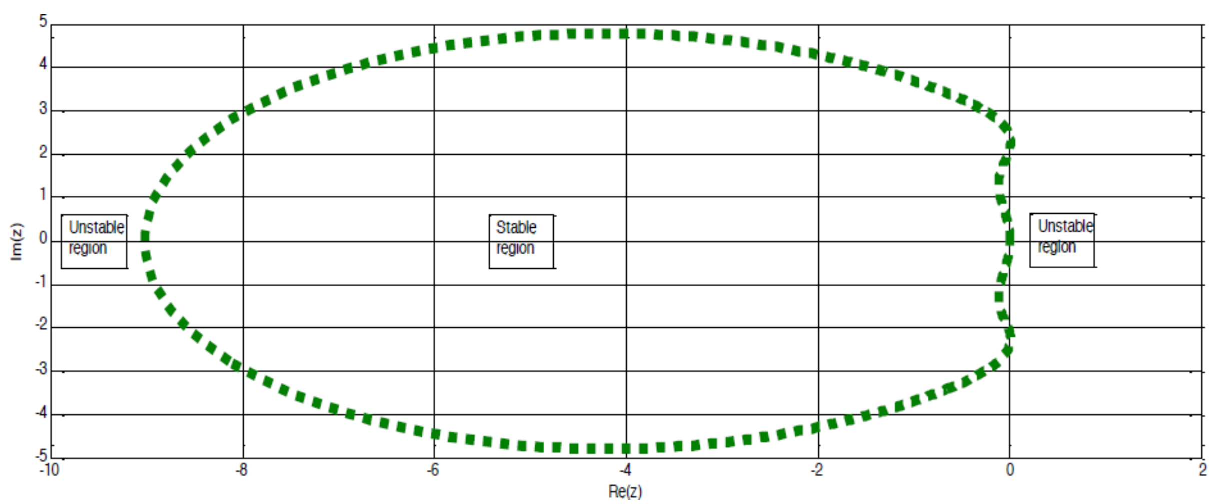

Fig.2.Stability regionof thethree-step hybridblock methodwith twooff-grid points.

Thestabilityregion obtainedin Fig.2 isalso A-stable.

IV. NUMERICALEXPERIMENTSA NDDISCUSSIONOFRESULTS

Mathematics-Based Experiments
Linear and nonlinear systems of equations of the type (1) should be solved using the newly developed three-step hybrid block techniques. A tabular presentation of the data will be provided based on the numerical experiments.

The tables below must adhere to the following notations:
3SHM1O-A three-stage hybrid block technique including a single off-grid junction
Two off-grid points in the 3SHM2O three-step hybrid block method First Issue

Considerthelinear stiffsystem in therange$0\leq x\leq 1$solved by [4],

$$\begin{bmatrix} y_1'(x) \\ y_2'(x) \end{bmatrix} = \begin{bmatrix} -1 & 95 \\ -1 & -97 \end{bmatrix} \begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix}, \begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad (24)$$

whoseexactsolutionisgiven by,

$$\begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix} = \frac{1}{47} \begin{bmatrix} 95e^{-2x} - 48e^{-95x} \\ 48e^{-96x} - e^{-2x} \end{bmatrix} \quad (25)$$

The eigenvalues of the Jacobian matrix are $\lambda_1=-2$, $\lambda_2=-96$ with the stiffness ratio 1:48

TABLEI:COMPARISONOFTHEABSOLUTEERRORSIN3SHM1OAND3SHM2O WITHTHATOF[4] FORPROBLEM 1

h	y_i	Error in 3SHM1O	Error in 3SHM2O	Error in [4]
0.0625	y_1	2.1998×10^{-15}	4.1562×10^{-14}	3×10^{-08}
	y_2	5.1273×10^{-13}	8.7325×10^{-16}	4×10^{-10}

Problem 1 was solved at the step size h = 0.0625 in order to compare our result with that of [4]. The results obtained clearly depict that both 3SHM1O and 3SHM2O performed better than thatof [4].

Problem2

Considerthelinear mildlystiffsystem,

$$\begin{bmatrix} y_1'(x) \\ y_2'(x) \end{bmatrix} = \begin{bmatrix} 998 & 1998 \\ -999 & -1999 \end{bmatrix} \begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix}, \begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad (26)$$

Theexact solutionofthelinear system ofequations aboveis given by,

$$\begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix} = \begin{bmatrix} 4e^{-x} - 3e^{-1000x} \\ -2e^{-x} + 3e^{-1000x} \end{bmatrix} \quad (27)$$

ItisimportanttostatethattheeigenvaluesoftheJacobianmatrixareλ_1=–1,λ_2=–1000withthe stiffness ratio 1: 1000.

TABLEII:COMPARISONOFTHEABSOLUTEERRORSIN3SHM1OAND3SHM2O WITH THAT OF [2] FOR PROBLEM 2

x	y_i	Error in 3SHM1O	Error in 3SHM2O	Error in [2]
5	y_1	1.2352×10^{-13}	2.1372×10^{-15}	1.3920×10^{-11}
	y_2	2.3527×10^{-13}	8.7325×10^{-16}	6.9700×10^{-12}
40	y_1	3.1562×10^{-16}	2.3372×10^{-18}	3.3628×10^{-12}
	y_2	3.1625×10^{-16}	4.1783×10^{-18}	1.6818×10^{-12}
70	y_1	4.2561×10^{-19}	3.2891×10^{-21}	3.9325×10^{-13}
	y_2	3.6172×10^{-20}	3.2891×10^{-22}	1.4664×10^{-13}

Problem3

Considerthewell-knownnonlineartwo-dimensionalKapsproblemintherange0$\leq$x$\leq$20 solved by [7],

$$\begin{bmatrix} y_1'(x) \\ y_2'(x) \end{bmatrix} = \begin{bmatrix} -1002y(x)+1000y^2(x) \\ y_1(x)-y_2(x)(1+y_2(x)) \end{bmatrix}, \begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad (28)$$

Theexactsolutionisgiven by,

$$\begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix} = \begin{bmatrix} e^{-2x} \\ e^{-x} \end{bmatrix} \quad (29)$$

The absolute error at the h the step size and N the number of computation steps are presented in Table III.

TABLEIII:COMPARISONOFTHEABSOLUTEERRORSIN3SHM1OAND3SHM2O WITH THAT OF [7] FOR PROBLEM 3

x	N	$y_!$	Error in 3SHM1O	Error in 3SHM2O	Error in [7]
5	4	y"	4.5627 × 10#""	1.6661 × 10#")	2.1670 × 10#&+
		y(	3.3316 × 10#&,	1.2112 × 10#"&	1.3507 × 10#&$
1.25	8	y"	4.5471 × 10#""	1.6514 × 10#")	2.3329 × 10#&+
		y(	3.3616 × 10#&,	1.3507 × 10#"&	2.8914 × 10#&$
0.833	12	y"	4.5317 × 10#""	1.6445 × 10#")	2.3078 × 10#&+
		y(	3.5112 × 10#&,	1.3722 × 10#"&	2.9695 × 10#&$
0.625	16	y"	4.5164 × 10#""	1.6317 × 10#")	2.2987 × 10#&+
		y(	3.4072 × 10#&,	1.3978 × 10#"&	2.9986 × 10#&$
0.5	20	y"	4.5090 × 10#""	1.6126 × 10#")	2.2948 × 10#&+
		y(	3.4611 × 10#&,	1.4155 × 10#"&	3.0115 × 10#&$

Problem4

Considerthenonlinearsystem solvedby [9],

$$\begin{bmatrix} y_1'(x) \\ y_2'(x) \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 998 & -999 \end{bmatrix} \begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix} + \begin{bmatrix} 2\sin x \\ 999(\cos x - \sin x) \end{bmatrix}, \begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \qquad (30)$$

Theexactsolutionisgiven by,

$$\begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix} = \begin{bmatrix} 2e^{-x} + \sin x \\ 2e^{-x} + \cos x \end{bmatrix} \qquad (31)$$

Theabsoluteerrorattheendpoint x □ 10and h thestep sizearepresentedin TableIV.

TABLEIV:COMPARISONOFTHEABSOLUTEERRORSIN3SHM1OAND3SHM2O WITH THAT OF [9] FOR PROBLEM 4

x	y_i	Error in 3SHM1O	Error in 3SHM2O	Error in [9]
0.25	y_1	1.15622×10^{-14}	3.52617×10^{-15}	4.50751×10^{-14}
	y_2	4.53625×10^{-14}	3.57261×10^{-15}	4.84057×10^{-14}
0.5	y_1	4.53627×10^{-14}	5.71256×10^{-15}	9.85878×10^{-14}
	y_2	4.55142×10^{-14}	5.75123×10^{-15}	9.81437×10^{-14}
1.0	y_1	5.12356×10^{-14}	6.66782×10^{-15}	9.45910×10^{-14}
	y_2	5.23781×10^{-14}	6.78312×10^{-15}	9.45792×10^{-14}
2.0	y_1	2.17827×10^{-14}	3.12367×10^{-15}	1.68310×10^{-13}
	y_2	2.21627×10^{-14}	3.34516×10^{-15}	1.68365×10^{-13}
4.0	y_1	3.51452×10^{-14}	4.78162×10^{-15}	2.21378×10^{-13}
	y_2	3.57726×10^{-14}	4.85263×10^{-15}	2.23044×10^{-13}
6.0	y_1	2.27812×10^{-14}	5.26735×10^{-15}	1.01363×10^{-13}
	y_2	2.28192×10^{-14}	5.52673×10^{-15}	1.01474×10^{-13}
8.0	y_1	2.90273×10^{-14}	7.73564×10^{-15}	1.93401×10^{-13}
	y_2	2.94152×10^{-14}	7.93544×10^{-15}	1.94650×10^{-13}
10.0	y_1	5.72863×10^{-14}	9.83644×10^{-15}	6.10623×10^{-13}
	y_2	5.55265×10^{-14}	9.99836×10^{-15}	6.09068×10^{-13}

A. DiscussionofResults

The two three-step hybrid block approaches generated from the findings shown in Tables I–IV are computationally trustworthy and efficient in addressing the linear and nonlinear problems.

rigid structure types (1). Compared to the three-step hybrid block approach with one off-grid point, the three-step method with two off-grid points(16) clearly performed better.You can observe that both approaches are A-stable from the stability areas found; refer to Figures 1 and 2.

V. CONCLUSION

This research has derived a pair of three-stephy hybrid block methods.Both linear and nonlinear systems of first-order differential equations were solved using the techniques that were developed. The findings showed that the derived approaches outperformed the ones we compared them against. Consistent, stable, zero-stable, and convergent procedures were also discovered.

References

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